

SUBJECT: MATHEMATICS

**PAPER: ADVANCED MATHEMATICAL
METHODS**

PAPER CODE:H 3051(PAPER III)

CLASS: MSc III SEMESTER

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Periodic function. A function $f(x)$ defined on a given domain is called of period T if

$$f(x+T) = f(x) = f(x+2T) = f(x+3T) = \dots$$

Example. Let $f(x) = \cos x$

then $f(x+2\pi) = \cos(x+2\pi) = \cos x$

$f(x+4\pi) = \cos(x+4\pi) = \cos x$

etc

i. $T = 2\pi$ here

ii $\cos x$ is a periodic function of period 2π .

Also $\sin x$ is a periodic function of period 2π as

$$\sin(x+2\pi) = \sin x = \sin(x+4\pi) = \dots$$

$\tan x$ is a periodic function of period π

as $\tan(x+\pi) = \tan x = \tan(x+2\pi) = \dots$

Even function. A function $f(x)$ defined on a given interval is called an even function if

$$f(-x) = f(x), \forall x \in \text{given interval}$$

ex $f(x) = \cos x, x \in]-\pi, \pi[$

then $f(x)$ is an even function as $f(-x) = f(x)$

i.e. $\cos(-x) = \cos x, \forall x \in]-\pi, \pi[$.

Odd function. A function $f(x)$ on a given interval is called an odd function if $f(-x) = -f(x)$, $\forall x \in \text{given interval}$.

example. $f(x) = \sin x$ is an odd function as $f(-x) = -f(x), \forall x \in \text{given interval}$.

We must remember some formulae as under:

$$(1) \int_0^{2\pi} -\sin nx = 0 \quad (2) \int_0^{2\pi} \cos nx dx = 0$$

$$(3) \int_{-\pi}^{\pi} -\sin nx dx = 0$$

$$(4) \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$(5) \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

Proof of (4) Let

$$I = \int e^{ax} \sin bx dx$$

$$I = \sin bx \cdot \frac{e^{ax}}{a} - \int b \cos bx \cdot \frac{1}{a} e^{ax} dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int \cos bx e^{ax} dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left[\cos bx \cdot \frac{e^{ax}}{a} - \int (-) b \sin bx \cdot \frac{e^{ax}}{a} dx \right]$$

$$I = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int \sin bx e^{ax} dx$$

$$= I$$

$$\left(1 + \frac{b^2}{a^2}\right) I = \frac{1}{a^2} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx$$

$$\left(\frac{a^2 + b^2}{a^2}\right) I = \frac{1}{a^2} e^{ax} [a \sin bx - b \cos bx]$$

$$I = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$\therefore \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

Similarly we can prove (5).

State and prove Euler's formula for Fourier series.

Statement, Let $f(x)$ be a periodic function of period 2π defined on $[0, 2\pi]$ then $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin nx + \dots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0, a_1, a_2, \dots, a_n, \dots, b_1, b_2, \dots, b_n$ are called Fourier constants. (or Fourier coefficients). And can be calculated as

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

Proof. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Integrating (1) with respect to x from

$x=0$ to $x=2\pi$, we have

$$\int_0^{2\pi} f(x) dx = \frac{1}{2} a_0 \int_0^{2\pi} dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos nx dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin nx dx$$

$$\Rightarrow \int_0^{2\pi} f(x) dx = \frac{1}{2} a_0 \cdot 2\pi + \sum_{n=1}^{\infty} a_n (0) + \sum_{n=1}^{\infty} b_n (0)$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx \quad \left[\begin{array}{l} \because \int_0^{2\pi} \cos nx dx = 0 \\ \text{and} \int_0^{2\pi} \sin nx dx = 0 \end{array} \right]$$

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Again, Integrating (1) after multiplying $\cos nx$ to both sides with respect to x from $x=0$ to $x=2\pi$, we have

$$\begin{aligned} \int_0^{2\pi} f(x) \cos nx \, dx &= \frac{1}{2} a_0 \int_0^{2\pi} \cos nx \, dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos^2 nx \, dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin nx \cos nx \, dx \\ \Rightarrow \int_0^{2\pi} f(x) \cos nx \, dx &= \frac{1}{2} a_0 (0) + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \left(\frac{1 + \cos 2nx}{2} \right) dx \\ &\quad + \sum_{n=1}^{\infty} b_n (0) \\ &= \sum_{n=1}^{\infty} a_n \cdot \frac{1}{2} \cdot 2\pi \end{aligned}$$

Comparing the coefficient of a_n , we have

$$\int_0^{2\pi} f(x) \cos nx \, dx = \pi a_n$$

or, $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$

Again, Integrating (1) after multiplying $\sin nx$ to both sides with respect to x from $x=0$ to $x=2\pi$, we have

$$\begin{aligned} \int_0^{2\pi} f(x) \sin nx \, dx &= \frac{1}{2} a_0 \int_0^{2\pi} \sin nx \, dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos nx \sin nx \, dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin^2 nx \, dx \\ &= \frac{1}{2} a_0 (0) + \sum_{n=1}^{\infty} a_n (0) + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \frac{1 - \cos 2nx}{2} dx \\ \int_0^{2\pi} f(x) \sin nx \, dx &= \sum_{n=1}^{\infty} b_n \cdot \pi \end{aligned}$$

Comparing the coefficient of b_n from both sides, we get

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

Proved.

⑤ Numericals on Fourier series

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Expand $f(x) = |x|$, Domain $[-\pi, \pi]$

Deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

We know $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^0 |x| dx + \frac{1}{2\pi} \int_0^{\pi} |x| dx$$

$$= \frac{1}{2\pi} \left[-\left(\frac{x^2}{2}\right)_{-\pi}^0 \right] + \frac{1}{2\pi} \left(\frac{x^2}{2} \right)_0^{\pi}$$

$$= \frac{1}{2\pi} \left[0 + \frac{\pi^2}{2} \right] + \frac{1}{2\pi} \left[\frac{\pi^2}{2} \right] = \frac{\pi}{2}$$

Note: $\int_{-\pi}^{\pi} f(x) dx = 2 \int_0^{\pi} f(x) dx$ if $f(x)$ is even
 i.e. $\frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{2}{2\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \cdot \frac{\pi^2}{2} = \frac{\pi}{2}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \quad \text{as } f(x) = |x| \cos nx$$

$$a_n = \frac{2}{\pi} \left[x \frac{\sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{\sin nx}{n} dx \right] \quad \begin{matrix} |x| \geq 0 \\ | -x | = |x| \end{matrix}$$

$$a_n = \frac{2}{\pi} \left[\pi \frac{\sin n\pi}{n} + \frac{1}{n^2} [\cos nx]_0^{\pi} \right]$$

$$= \frac{2}{\pi} \left[0 + \frac{1}{n^2} (\cos n\pi - 1) \right] \quad [\because \sin n\pi = 0]$$

$$a_n = \frac{2}{\pi n^2} [(-1)^n - 1] = \dots$$

(i) if n is even $a_n = 0$

(ii) if n is odd $a_n = -\frac{4}{\pi n^2}$ i.e. $a_1 = -\frac{4}{\pi \cdot 1^2}, a_3 = -\frac{4}{\pi \cdot 3^2}, \dots$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx dx = 0 \quad \text{as } \int_{-\pi}^{\pi} f(x) dx = 0 \quad \text{if } f(x) \text{ is odd}$$

$$|x| = f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos nx + 0$$

$$|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[-\frac{4}{\pi n^2} \cos nx \right]_{n \text{ is odd}} + 0$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{1}{1^2} \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right]$$

Put $x=0$

$$\frac{\pi}{2} = \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\text{OR, } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\underline{\underline{2}} \quad \text{Expand } f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & \text{if } x=0 \\ 1, & \text{if } 0 < x < \pi \end{cases}$$

$$\text{also deduce } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{2\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 (-1) dx + \frac{1}{2\pi} \int_0^{\pi} 1 dx$$

$$= -\frac{1}{2\pi} [x]_{-\pi}^0 + \frac{1}{2\pi} [x]_0^{\pi} = -\frac{1}{2} + \frac{1}{2} = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-1) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} 1 \cos nx dx$$

$$= 0 - \frac{1}{\pi} \int_{-\pi}^0 \sin nx dx + \frac{1}{\pi} \int_0^{\pi} \sin nx dx$$

$$= \frac{1}{\pi n} [\cos nx]_{-\pi}^0 - \frac{1}{\pi n} [\cos nx]_0^{\pi}$$

$$= \frac{1}{\pi n} [1 - \cos n\pi] - \frac{1}{\pi n} [\cos n\pi - 1]$$

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$$b_n = \frac{2}{n\pi} [1 - \cos n\pi]$$

$$= \frac{2}{n\pi} [1 - (-1)^n]$$

$$(-1)^n = 1 \text{ if } n \text{ is even}$$

$$(-1)^n = -1 \text{ if } n \text{ is odd}$$

$$b_2 = b_4 = 0 = b_6 = \dots$$

$$b_1 = \frac{4}{\pi}, b_3 = \frac{4}{3\pi}, b_5 = \frac{4}{5\pi}, \dots$$

$$i. f(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \dots$$

$$\text{at } x = \frac{\pi}{2}, f(x) = 1$$

$$\sin \frac{3\pi}{2} = \sin(\pi + \frac{\pi}{2})$$

$$= -\sin \frac{\pi}{2}$$

$$1 = \frac{4}{\pi} \left(\frac{4}{3\pi} + \frac{4}{5\pi} + \dots \right)$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

$$\sin \frac{5\pi}{2} = \sin \frac{\pi}{2}$$

$$\sin \frac{7\pi}{2} = -1$$

$$\begin{cases} \sin(\text{odd}\pi + \frac{\pi}{2}) = -1 \\ \sin(\text{even}\pi + \frac{\pi}{2}) = 1 \end{cases}$$

③ Show that an even fn can have

no sine term in its Fourier series $f(x) = x \sin x$,

on $[-\pi, \pi]$, deduct $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \dots$

we have Fourier series as

$$\text{Sol } f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin x dx$$

$$a_0 = \frac{1}{2\pi} \cdot 2 \cdot \int_0^{\pi} x \sin x dx \quad \text{as } (x \sin x) \text{ is even fn}$$

$$a_0 = \frac{1}{\pi} \left[-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx \right] \quad \left[\because 2 \cos A \sin B = \sin(A+B) - \sin(A-B) \right]$$

$$a_0 = \frac{1}{\pi} [+\pi + 0] = 1$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos nx dx = \frac{x}{\pi} \int_0^{\pi} \frac{1}{2} \cdot 2 \cos nx \sin x \cdot x dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \cdot [\sin(n+1)x - \sin(n-1)x] dx$$

$$a_n = \frac{1}{\pi} \left[x \left\{ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} \right]_0^\pi - \int_0^\pi \left(\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1} \right) dx$$

$$a_n = \frac{1}{\pi} \left[\pi \left\{ \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right\} \right] - 0$$

$$a_n = \frac{1}{n-1} (-1)^{n-1} - \frac{1}{n+1} (-1)^{n+1}, \quad n \geq 2$$

$$a_2 = \frac{1}{3} - 1, \quad a_3 = \frac{1}{2} (-1)^2 - \frac{1}{4} (-1)^4 = \frac{1}{2} - \frac{1}{4}$$

$$a_4 = \frac{1}{3} (-1)^3 - \frac{1}{5} (-1)^5 = -\frac{1}{3} + \frac{1}{5}$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \frac{\sin 2x}{2} \, dx = \frac{1}{\pi} \left[x \left(-\frac{\cos 2x}{2} \right) + \int_0^{\pi} \frac{\cos 2x}{2} \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi (-1) \cos 2\pi}{2} \right] = -\frac{1}{2}$$

$$\text{Now, } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \sin nx \, dx$$

$$\begin{aligned} \text{Let } g(x) &= x \sin x \sin nx \\ g(-x) &= -x \sin(-x) \sin(-nx) \\ &= -x \sin x \sin nx \\ &= -g(x) \end{aligned}$$

$$\therefore b_n = \frac{1}{\pi} \times 0 = 0$$

i.e. $g(x)$ is odd fcn

$$\therefore x \sin x = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \dots$$

$$x \sin x = 1 - \frac{1}{2} \cos 2x - \frac{2}{3} \cos 3x + \frac{1}{4} \cos 4x - \frac{2}{15} \cos 5x + \dots$$

$$\text{Put } x = \frac{\pi}{2}$$

$$\frac{\pi}{2} \sin \frac{\pi}{2} = 1 - \frac{1}{2} \times 0 - \frac{2}{3} \cos \pi + \frac{1}{4} \cos \frac{3\pi}{2} - \frac{2}{15} \cos \frac{5\pi}{2} + \dots$$

$$\frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \dots \Rightarrow \frac{\pi}{2} = 2 \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \dots \right]$$

$$\frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \dots \Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \dots$$

4. Expand $f(x) = \frac{\pi-x}{2}$, $0 < x < 2\pi$ in a Fourier series. Also deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Soln. $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\pi}{2} - \frac{x}{2} \right) dx \\ &= \frac{1}{2\pi} \times \frac{\pi}{2} \int_0^{2\pi} dx - \frac{1}{4\pi} \int_0^{2\pi} x dx \\ &= \frac{1}{4} \cdot 2\pi - \frac{1}{4\pi} \cdot \frac{1}{2} [x^2]_0^{2\pi} \\ &= \frac{\pi}{2} - \frac{\pi}{2} = 0. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} \cdot \cos nx dx \\ &= \frac{1}{\pi} \times \frac{\pi}{2} \int_0^{2\pi} \cos nx dx - \frac{1}{2\pi} \int_0^{2\pi} x \cos nx dx \\ &= \frac{1}{2} \int_0^{2\pi} \cos nx dx - \frac{1}{2\pi} \left[x \cdot \frac{\sin nx}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin nx}{n} dx \\ &= 0 - \frac{1}{2\pi} \left[0 - \frac{1}{n} \int_0^{2\pi} \sin nx dx \right] \\ &= \frac{1}{2\pi n} \int_0^{2\pi} \sin nx dx = \frac{1}{2\pi n^2} [\cos nx]_0^{2\pi} = 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} \frac{\pi-x}{2} \cdot \sin nx dx \\ &= \frac{1}{2} \int_0^{2\pi} \sin nx dx - \frac{1}{2\pi} \int_0^{2\pi} x \sin nx dx \\ &= -\frac{1}{2} \left[\frac{\cos nx}{n} \right]_0^{2\pi} - \frac{1}{2\pi} \left[-x \frac{\cos nx}{n} \right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos nx}{n} dx \\ &= 0 + \frac{1}{2\pi n} [2\pi \cos 2n\pi] - \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos nx}{n} dx = \frac{1}{n} \end{aligned}$$

$$i) a_0 = 0, a_n = 0, b_n = \frac{1}{n}$$

$$ii) \frac{\pi - x}{2} = 0 + 0 + \sum \frac{1}{n} \sin nx$$

$$\frac{\pi - x}{2} = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{4} \sin 4x + \dots$$

$$\text{Put } x = \frac{\pi}{2}$$

$$\frac{\pi}{4} = 1 + 0 + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}1 + \frac{1}{6} \times 0 + \frac{1}{7}(-1) + \dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

⑤ Obtain the Fourier series of $f(x) = x^2$ in $[-\pi, \pi]$ and show that

$$(i) \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

$$(ii) \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots$$

$$(iii) \frac{\pi^2}{48} = \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \dots$$

$$(iv) \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$$

Sol we have F. series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{3\pi} \pi^3 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - \int_0^{\pi} 2x \frac{\sin nx}{n} dx \right]$$

$$= -\frac{4}{\pi n} \int_0^{\pi} x \sin nx dx$$

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$$a_n = -\frac{4}{\pi n} \int_0^{\pi} x \sin nx dx$$

$$= -\frac{4}{\pi n} \left[-x \frac{\cos nx}{n} + \int_0^{\pi} \frac{\cos nx}{n} dx \right]$$

$$= \frac{4}{\pi n^2} \left[x \cos nx \right]_0^{\pi} - \frac{4}{\pi n^2} \int_0^{\pi} \cos nx dx = 0$$

$$= \frac{4}{\pi n^2} \left[\pi \cos n\pi \right]$$

$$a_n = \frac{4}{\pi n^2} \left[\pi (-1)^n \right] = \frac{4}{n^2} (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx dx = 0 \text{ as } (x^2 \sin nx) \text{ is odd fcn}$$

$$i. x^2 = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx + 0$$

$$x^2 = \frac{\pi^2}{3} - \frac{4 \cos x}{1^2} + \frac{4 \cos 2x}{2^2} - \frac{4 \cos 3x}{3^2} + \frac{4 \cos 4x}{4^2} - \dots$$

$$x^2 - \frac{\pi^2}{3} = -4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots \right]$$

$$(i) \text{ Put } x = \pi$$
$$\pi^2 - \frac{\pi^2}{3} = -4 \left[\frac{\cos \pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \dots \right]$$

$$\frac{2\pi^2}{3} = -4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$(ii) \text{ Put } x = 0$$
$$-\frac{\pi^2}{3} = -4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

(12) (11-8651), by Shishu Kumar

(iii) Let $x = \frac{\pi}{2}$

$$\begin{aligned} \frac{\pi^2}{4} - \frac{\pi^2}{8} &= -\frac{1}{2} \left[\frac{1}{1^2} \cos \frac{\pi}{2} - \frac{1}{2^2} \cos 2 \cdot \frac{\pi}{2} + \frac{1}{3^2} \cos 3 \cdot \frac{\pi}{2} \right. \\ &\quad \left. - \frac{1}{4^2} \cos 4 \cdot \frac{\pi}{2} + \dots \right] \\ \frac{-\pi^2}{12} &= -\frac{1}{2} \left[0 + \frac{1}{2^2} - \frac{1}{3^2} \times 0 - \frac{1}{4^2} + \dots \right] \\ \frac{\pi^2}{24} &= \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \dots \end{aligned}$$

(iv) Adding (i) and (ii), we get

$$\begin{aligned} \frac{\pi^2}{6} + \frac{\pi^2}{12} &= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right) \\ &\quad + \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) \\ \frac{5\pi^2}{12} &= 2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \\ \frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \end{aligned}$$

6 Find Fourier Series of

$$f(x) = \begin{cases} x - \pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

We have

$$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{2\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 (x - \pi) dx + \frac{1}{2\pi} \int_0^{\pi} (\pi - x) dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 x dx - \frac{1}{2} \int_{-\pi}^0 dx + \frac{1}{2} \int_0^{\pi} dx - \frac{1}{2\pi} \int_0^{\pi} x dx$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2} [x^2]_{-\pi}^0 - \frac{1}{2} [x]_{-\pi}^0 + \frac{1}{2} [x]_0^{\pi} - \frac{1}{2\pi} \cdot \frac{1}{2} [x^2]_0^{\pi}$$

$$= \frac{1}{4\pi} [0 - \pi^2] - \frac{1}{2} [0 + \pi] + \frac{1}{2} [\pi] - \frac{1}{4\pi} [\pi^2]$$

$$= -\frac{\pi}{4} - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = -\frac{2\pi}{4} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 (x - \pi) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 x \cos nx dx - \int_{-\pi}^0 \cos nx dx + \int_0^{\pi} \cos nx dx - \frac{1}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} \Big|_{-\pi}^0 - \int_{-\pi}^0 \frac{\sin nx}{n} dx \right] - 0 + 0 - \frac{1}{\pi} \left[x \frac{\sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{n} \int_{-\pi}^0 \sin nx dx \right] - \frac{1}{\pi n} \left[- \int_0^{\pi} \sin nx dx \right]$$

$$= + \frac{1}{\pi n^2} [\cos nx]_{-\pi}^0 - \frac{1}{\pi n^2} [\cos nx]_0^{\pi} = \frac{1}{\pi n^2} [1 - (-1) - (-1) + 1]$$

(14) H-3051, Dr. Satish Kumar.

$$a_n = \frac{2}{\pi n^2} [1 - (-1)^n]$$

$$a_n = \frac{2}{\pi n^2} [1 + (-1)^{n+1}]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^0 (x - \pi) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 x \sin nx \, dx - \int_{-\pi}^0 \sin nx \, dx + \int_0^{\pi} \sin nx \, dx - \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi} \left[-\frac{x \cos nx}{n} \right]_{-\pi}^0 + \left[\frac{\cos nx}{n} \right]_{-\pi}^0 + \frac{1}{n} (\cos nx) \Big|_{-\pi}^0 - \frac{1}{n} [\cos nx] \Big|_0^{\pi} - \frac{1}{\pi} \left[-\frac{x \cos nx}{n} \right]_0^{\pi} + \left[\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[0 - \frac{\pi \cos \pi n}{n} + \frac{1}{n} (0) \right] + \frac{1}{n} [1 - (-1)^n] - \frac{1}{n} [(-1)^n - 1]$$

$$+ \frac{1}{\pi n} [\pi \cos n\pi] - \frac{1}{\pi} \times 0$$

$$= -\frac{1}{n} (-1)^n + \frac{1}{n} - \frac{1}{n} (-1)^n - \frac{1}{n} (-1)^n + \frac{1}{n} + \frac{1}{n} (-1)^n$$

$$= \frac{2}{n} - \frac{2}{n} (-1)^n$$

$$b_n = \frac{2}{n} [1 - (-1)^n] = \frac{2}{n} [1 + (-1)^{n+1}]$$

$$ii. f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{1 + (-1)^{n+1}}{n^2} \right] \cos nx + \sum_{n=1}^{\infty} \frac{2}{n} [1 + (-1)^{n+1}] \sin nx$$